Calculation of Heat Flow in a Medium the Conductivity of which Varies with Temperature

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It is shown that the heat flow in a single homogeneous medium of thermal conductivity $\sigma(T)$, between boundaries at temperatures T_2 and T_1 is the same as that for the same geometry in a medium of unit thermal conductivity and boundary "thermal potentials" $\Sigma(T_2)$ and $\Sigma(T_1)$, where $\Sigma(T)=\int_0^T \sigma(T')dT'$. Some applications are given, as well as a short table of $\Sigma(T)$ for common cryogenic temperatures and materials.

IT is the purpose of this note to present a theorem[1] which is quite useful in the design of cryogenic apparatus and may also have other applications. One often has the problem of calculating the flow of heat, say down an inconel tube which joins a copper can at 78°K to one at 4°K. The thermal conductivity of inconel is, however, a strong function of temperature in this range and it is not at first clear what average conductivity to take, or how this average depends upon the geometry. We give here a simple proof of a theorem which shows that, not only in this one-dimensional case but also in the general three-dimensional case of a medium with boundaries at two temperatures, the total heat transfer may be computed as if the medium had a conductivity which is just a linear average of the temperature-varying conductivity over the temperature interval. In particular we shall prove the

Theorem: for a homogeneous medium with heat conductivity $\sigma=\sigma(T)$ the heat flow between two boundaries of the medium maintained at temperatures T_1 and T_2 is

$$H=F(G)(T_2-T_1)\bar{\sigma}=F(G)[\Sigma(T_2)-\Sigma(T_1)], \quad (1)$$

where

$$\bar{\sigma}(T_1,T_2)=\frac{\Sigma(T_2)-\Sigma(T_1)}{T_2-T_1},$$

$$\Sigma(T)=\int_0^T \bar{\sigma}(T')dT', \quad (2)$$

and $F(G)$ signifies a function of geometry only, having the dimensions of length.[2] This means that the heat-flow problem is not essentially changed by the introduction of the varying conductivity and that for calculations between two given temperatures the problem reduces to finding the average conductivity between these two temperatures.

Let us consider the problem of calculating the magnitude of the heat flow in a poor conductor, with boundaries at greatly differing temperatures (T_1 and T_2, say). Figure 1(a) shows the case in which the conductivity σ is independent of temperature; Fig. 1(b) shows a cylindrical conductor with varying conductivity $\sigma(T)$; Fig. 1(c) a variable-area, variable-conductivity conductor; and Fig. 1(d) is Fig. 1(c) with T_2 and T_1 interchanged. Intuitively by application of first-order perturbation theory one might think that, if $\sigma(T)$ were small for T near T_1 and large for T near T_2, then the heat flow in Fig. 1(c) would be greater than that for Fig. 1(d). It will be shown, however, that the heat flows in Figs. 1(c) and 1(d) are the same. It will be shown also that the heat transfer between the boundaries at T_2 and T_1 for $\sigma(T)$ of the form of Fig. 2(a) is very much larger than that for Fig. 2(b), which latter is about the same as that for Fig. 2(c), perhaps contrary to expectation.

To determine the heat flow we must solve the diffusion equation for the steady state,

$$\nabla\cdot[\sigma(T)\nabla T]=0. \quad (3)$$

The heat current $\mathbf{h}$ is given by

$$\mathbf{h}=\sigma(T)\nabla T. \quad (4)$$

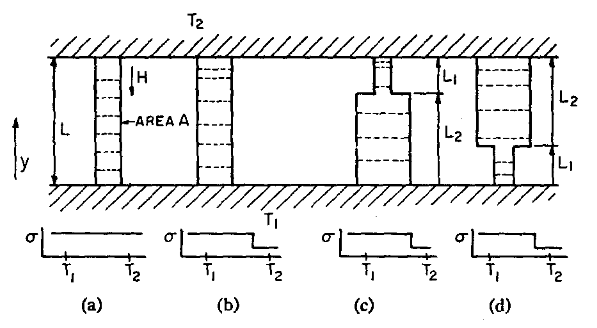

FIG. 1. Heat flow H for various geometries and conductivities $\sigma(T)$. Isotherms are sketched (dotted) for equal temperature intervals.

$$\text{In}(a),\ H_a=(T_2-T_1)\frac{\sigma A}{L};$$

$$\text{In}(b),\ \frac{dT}{dy}=\frac{H_b}{A\sigma(T)},\quad \sigma(T)dT=\frac{H_b}{A}dy$$

$$H_b=\frac{A}{L}\int_{T_1}^{T_2}\sigma(T)dT;$$

$$\text{In}(c),\ H_c=?;$$

$$\text{In}(d),\ H_d=?.$$

[1] The theorem is apparently due to Kirchoff, but does not appear in standard works in heat transmission and is evidently almost unknown to both physicists and mechanical engineers. In the past, considerable unnecessary effort has been expended on numerical work, which can be eliminated by the use of the theorem and a table of thermal potentials $\Sigma(T)$. The same theorem is proved by H. G. Elrod, Jr., Trans. Am. Soc. Mech. Engrs. **70**, 905 (1948), but it appears still unknown to cryogenicists.

[2] As remarked by P. J. Price, the theorem is true also for tensor conductivity, the magnitude of which is a function of temperature.

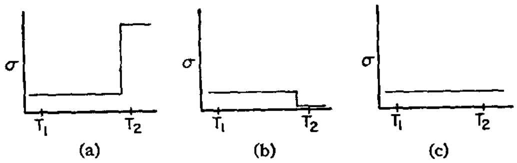

FIG. 2. Some possible thermal conductivities $\sigma(T)$. The zero conductivity region in "b" has little influence on the heat flow between T_2 and T_1.

Let us define a function

$$\Sigma(T)=\int_0^T \sigma(T')dT', \tag{5}$$

which may be called the "thermal potential." Now

$$\nabla\Sigma(T)=\frac{d\Sigma(T)}{dT}\nabla T=\sigma(T)\nabla T=\mathbf{h}. \tag{6}$$

The gradient of $\Sigma(T)$ is therefore the heat current $\mathbf{h}$, the divergence of which is zero, so that Σ is required to satisfy the Laplace equation. No matter what the form of $\sigma(T)$, the isothermals retain the same shape, and along an isothermal $\sigma(T)$ is constant; so the "isothermals" of $\Sigma(T)$ coincide with those of T itself.

$$\nabla^2\Sigma(T)=0. \tag{7}$$

Evidently $\Sigma(T)$ may be regarded as the "temperature," in the sense that we may substitute the problem of the same geometry G with boundaries at "temperature" $\Sigma(T_2)$ and $\Sigma(T_1)$. Since the gradient of Σ itself is the heat flow, the thermal conductivity in this new problem must be regarded as unity. Therefore, the total flow of heat between the boundaries at temperature T_2 and temperature T_1 is given by

$$\begin{aligned}
&\text{(a)}\quad H=[\Sigma(T_2)-\Sigma(T_2)]F(G),\\
&\text{(b)}\quad \;\;=F(G)\left[\int_0^{T_2}\sigma(T')dT'-\int_0^{T_1}\sigma(T')dT'\right],\\
&\text{(c)}\quad \;\;=F(G)\int_{T_1}^{T_2}\sigma(T')dT',\\
&\text{(d)}\quad \;\;=F(G)(T_2-T_1)\bar{\sigma}(T_2,T_1),
\end{aligned} \tag{8}$$

so that two situations which differ from one another only in conductivities $\sigma(T)$ will have relative heat transfers proportional to the respective values of $\bar{\sigma}$ in the two cases.

No such theorem will hold, of course, if the isothermals are distorted under $\sigma\rightarrow\sigma(T)$ such as in a composite medium where $\sigma=\sigma(T,\mathbf{r})$. On the other hand, a combination of a homogeneous medium with some perfectly-conducting or perfectly-insulating regions presents no difficulty. Thus we have established the theorem.

Some corollaries of the theorem appear:

(I) Under the conditions assumed (σ a function of temperature only, the diffusion equation valid for heat flow by virtue of the phonon mean free path being small compared to dimensions of the problem), the heat transfer between two boundaries cannot decrease with decrease in temperature of the cold boundary, nor with increase in temperature of the hot. Thus, even if $\sigma(T)=0$ for $0<T<4$°K and $\sigma(T)=(T-4°)$ for $T>4$°K, the heat flow from 78°K to 1°K cannot be less than the heat flow from 78° to 4°. In fact, in this case the heat flow is unchanged no matter what the colder boundary temperature is below 4°K. Thus a small region of low conductivity in the curve of σ *vs* T is not important in designing cryogenic apparatus, while a small range of T in which $\sigma(T)$ is very large would mean a very large heat influx since the thermal potential is the area under the $\sigma(T)$ curve.

(II) Accordingly, we need only a table of values of $\Sigma(T)=\int_0^T\sigma(T')dT'$ to calculate the flow of heat between two boundaries in any homogeneous medium. An example is presented below as Table I. One must be sure, of course, in applying the theorem that radiation or conduction from the "free" boundaries is negligible. With these precautions, the theorem is of considerable use in heat flow computations.

TABLE I. Thermal potentials $\Sigma(T)$.[a]

Material \ T°K	4.2°K	78°K	300°K
Constantan	16	9.9×10^3	5.6×10^4
Inconel (drawn)	5.4	3.8×10^3	3.3×10^4
Stainless steel (347)	4.7	3.3×10^3	3.0×10^4
Copper (W-2, Fig. 11)	4.4×10^4	1.7×10^6	2.7×10^6

[a] Thermal conductivities taken from NBS Circular 556, (1954). Large variations from sample to sample must be expected. $\Sigma(T)$ in units of 10^{-3} watts/cm.

(III) Although heat flow in a single medium with $\sigma(T)$ is not a linear phenomenon, it is reversible in that an interchange of boundary temperatures leads to the same heat flux. This statement definitely does not hold where σ is a function of position for fixed temperature since the differential Eq. (3) then does not separate.

As an example of the value of the theorem we calculate now the heat flow for Fig. 1(c) it is given by

$$H\left(\frac{L_1}{A_1}+\frac{L_2}{A_2}\right)=\Delta\Sigma,\quad H=[\Sigma(T_2)-\Sigma(T_1)]\frac{A_1A_2}{A_1L_2+A_2L_1}.$$

The identical value is obtained for Fig. 1(d).

As an example of a case with σ a rapidly varying function of T, we may take the transfer of heat by radiation in the interior of a star in which local near equality of radiation and matter temperatures exist. Here we have

$$\nabla\cdot[\bar{\lambda}(T)\nabla(T^4)]=0, \tag{14}$$

where $\bar{\lambda}$ is the Rosseland mean free path. The heat flux is now $\mathbf{h}\alpha\bar{\lambda}\nabla(T^4)$. If the question is put whether a star of uniform matter distribution, but with an opacity which is an arbitrary (positive) function of temperature, can insulate itself by keeping its surface cool, or whether the heat transfer must increase with decreasing surface temperature; the answer is evidently that the latter is true, since all contributions to the integral of Eq. 8(c) are positive. (Here $\sigma(T)=4T^3\bar{\lambda}(T)$.) Therefore even if heat transfer is locally by radiation, the theorem holds and the heat flow can only increase as the boundary temperature decreases.

On the other hand it is entirely possible for heat transfer *in general* to decrease with decrease in temperature of the cold boundary. A familiar example is a well-silvered dewar, in which because of the increased conductivity and reflectivity of silver at low temperature, the heat absorbed from a surface at 300°K by a silver wall at 4°K may be only one-half of the heat absorbed by a wall at 78°K. This appears to violate the theorem proved in the foregoing, since the heat flow decreases with increasing temperature span, but there is in this case no local equilibrium, the heat transfer is not governed by Fourier's equation, and the situation is not to be described as above. As in all cases one must clearly separate the validity of the theorem from the pertinence of the assumptions.

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Direct Method of Measuring the Contact Injection Ratio*

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A direct method of measuring the injection ratio of rectifying contacts is described. With this method, the injection ratio is determined by comparing the area of a current injection pulse with the area of the resulting hole storage pulse as observed on an oscilloscope screen.

I. INTRODUCTION

THE purpose of this paper is to report a direct method of measuring the injection ratio[1] of a rectifying barrier, which requires no auxiliary contacts. Previous methods for determining the injection ratio either require auxiliary contacts for a direct measurement,[2] or are indirect.[3–5] The method entails comparing the areas of two pulses as observed on an oscilloscope. These areas give the time integrals of a current injection pulse and the resulting hole-storage pulse.

The term "hole storage" was originated by Michaels and Meacham.[6] It refers to the holes, injected in n-type material by an emitting contact, which are swept back to the emitter by reverse bias, and thereby produce a hole-storage pulse. Studies of the hole-storage effect have been concentrated on two different groups of experiments conducted for the most part on diffused junctions (for which $\gamma=1$). One group of experiments considers the reverse recovery current after a pulse of forward current,[7–11] and the other considers the open circuit voltage after a pulse of forward current.[12–17] The area of the hole storage pulse [area A_2, Fig. 1(b)] shrinks because of hole-electron recombination. We found that it was necessary to apply a fixed reverse bias [Fig. 1(a)] in order to minimize this shrinkage, and a bias of a few tenths of a volt was found to be suitable. (There is a hole-storage pulse without applied bias because of the post-injection emf.[12]) Values of injection ratio in the range $0.2<\gamma<1.0$ were measured.

II. ANALYSIS AND DISCUSSION

By applying an input pulse [Fig. 1(a)] with sufficient amplitude compared with the reverse bias, the diode is able to draw forward current and holes are injected into the semi-conducting region nearby. After the pulse the electrons which have entered the metal cannot return to the semiconductor because they encounter the metal-semiconductor barrier. We assume, following Swanson,[18] that the electronic barrier from metal to semiconductor

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1 The injection ratio is defined as the fraction of the dc current carried by minority carriers, and is denoted herein by the symbol γ. This discussion refers to n-type material. Thus the minority carriers are holes.

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